

A GENERALIZATION OF THE TRANSFER MAP IN THE COHOMOLOGY OF GROUPS⁽¹⁾

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1. **Introduction.** In this paper a multiplicative object analogous to transfer is defined and its properties discussed. The original motivation for defining such an object was to give relatively precise information about nontriviality for restriction and related questions as discussed by Swan [7] and the author [4, §7]. In addition we find ourselves led to interesting formulas relating this object to the Chern classes of representations as defined by Atiyah [1] and others. This relation will be the subject of a future paper.

2. **Wreath products and the monomial representation.** Let G be a group and H a subgroup of finite index l . Then there is a natural construction (going back to Frobenius) whereby G may be imbedded in the wreath product of the symmetric group \mathcal{S}_l with H . This is described below.

We shall use the notation of Hall [5, §5.9] concerning wreath products *except* that for us permutations will compose as functions and act on the left. Thus if S_i is a permutation group on the set $L_i = \{1, 2, \dots, l_i\}$ ($i = 1, 2$), then $S_1 \wr S_2$ is the group of permutations θ on $L_1 \times L_2$ satisfying

$$\theta(i, j) = (\sigma_i, \tau_i j)$$

where

$$\sigma \in S_1, \tau_i \in S_2 \quad (i = 1, 2, \dots, l_1).$$

Also, ordering $L_1 \times L_2$ lexicographically, we may identify it with $L = \{1, 2, \dots, l_1 l_2\}$ so that $S_1 \wr S_2$ may be viewed as a subgroup of $\mathcal{S}_{l_1 l_2}$.

If S is a permutation group of degree l and H is an abstract group then we have $S \wr H$, a permutation group on $L \times H$ where H acts on itself by left translation. In fact $S \wr H$ is the semi-direct product of S with H^l subject to

$$\pi^{-1} \sigma_1 \times \sigma_2 \times \dots \times \sigma_l \pi = \sigma_{\pi(1)} \times \sigma_{\pi(2)} \times \dots \times \sigma_{\pi(l)}$$

for $\pi \in S$ and $\sigma_i \in H$.

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Let $\tau_1, \tau_2, \dots, \tau_l$ be a left transversal of H in G . For $\rho \in G$ we have

$$\rho\tau_i = \tau_{\pi(i)}\sigma_i$$

for uniquely determined $\pi \in \mathcal{S}_l$, $\sigma_i \in H$.

Define

$$\Phi(\rho) = \pi \cdot \sigma_1 \times \sigma_2 \times \dots \times \sigma_l.$$

Then $\Phi: G \rightarrow \mathcal{S}_l \int H$ is a monomorphism.

If $\tau'_1, \tau'_2, \dots, \tau'_l$ is another transversal,

$$\tau'_i = \tau_{\alpha(i)}\beta_i$$

where $\alpha \in \mathcal{S}_l$ and $\beta_i \in H$. Letting

$$(1) \quad \mu = \alpha \cdot \beta_1 \times \beta_2 \times \dots \times \beta_l$$

we see that Φ' defined from the second transversal is related to Φ by

$$\Phi' = C_\mu \circ \Phi.$$

(Here $C_\mu(\gamma) = \mu^{-1}\gamma\mu$ for $\mu \in \mathcal{S}_l \int H$.)

3. **The monomial module.** Let A be a left H -module; we construct a corresponding $\mathcal{S}_l \int H$ -module as follows. Set

$$A^l = A \otimes_z A \otimes_z \dots \otimes_z A \quad (l \text{ times})$$

and define the operation of $\mathcal{S}_l \int H$ on this group by the formulas

$$(\sigma_1 \times \sigma_2 \times \dots \times \sigma_l)(a_1 \otimes \dots \otimes a_l) = \sigma_1 a_1 \otimes \sigma_2 a_2 \otimes \dots \otimes \sigma_l a_l,$$

$$\pi(a_1 \otimes a_2 \otimes \dots \otimes a_l) = a_{\pi^{-1}(1)} \otimes a_{\pi^{-1}(2)} \otimes \dots \otimes a_{\pi^{-1}(l)},$$

for $\sigma_i \in H$ and $\pi \in \mathcal{S}_l$. More generally if C is an H -complex in the sense of [3], then by introducing the appropriate sign C^l is made an $\mathcal{S}_l \int H$ -complex. (See [4, §5, p. 231].)

Suppose now that G is a group containing H as a subgroup of finite index l . By means of the injection Φ we may view A^l as a G -module which we call the monomial G -module induced from the H -module A and denote by $\mathcal{M}_{H \rightarrow G}(A)$. Naturally this definition depends on the choice of coset representatives. If $\tau'_1, \tau'_2, \dots, \tau'_l$ is another transversal, it yields another injection Φ' and another structure \mathcal{M}' of G -module on A^l . Then in fact the action of the element μ (formula (1)) on A^l yields a mapping

$$\mu: \mathcal{M}' \rightarrow \mathcal{M}$$

which is an isomorphism of G -modules.

As an example let $A = Z \oplus B$ where B is an H -module. Then

$$\mathcal{M}_{H \rightarrow G}(A) = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_l$$

where $\mathcal{M}_0 \cong Z$, $\mathcal{M}_1 \cong Z(G) \otimes_H B$ (the induced module), and $\mathcal{M}_l \cong \mathcal{M}_{H \rightarrow G}(B)$.

4. The norm map. Preliminaries on wreath products. In this section we outline the properties of a special case of the wreath product of cohomology classes which has been defined generally by Nakaoka.

Suppose first that C is a cochain complex (i.e., a complex involving only non-negative degrees). As in §3 make C^l an \mathcal{S}_l -complex.

Let S be a subgroup of \mathcal{S}_l and let W be an S -projective resolution of Z . We wish to study the *double complex* $D = \text{Hom}_S(W, C^l)$ ⁽²⁾. The homology of D (and also its other invariants [3, Chapter XV, §6]) is independent of the particular projective resolution W chosen. Also if $F: C \rightarrow C'$ is a map of complexes of degree zero then F induces

$$\text{Hom}_S(1, F^l) : \text{Hom}_S(W, C^l) \rightarrow \text{Hom}_S(W, C'^l),$$

a map of double complexes. By a lemma of Steenrod [6, Lemma 3.2], homotopic maps of C into C' induce homotopic (but *not* bi-homotopic) maps of the corresponding (double) complexes. Thus the homology of $\text{Hom}_S(W, C^l)$ depends only on the homotopy type of the complex C .

Following Borel and Hirzebruch [2], denote by $H^{**}(C)$ the direct product of the groups $H^n(C)$. (Then if C is also a differential graded ring, $H^{**}(C)$ has a resulting ring structure.)

Consider in particular the complex $Z[T]$ (T an indeterminate) where the grading is by degree and the differentiation is trivial. Then C^l may be identified with the polynomial ring $Z[T_1, T_2, \dots, T_l]$ on which \mathcal{S}_l acts by permuting the variables and multiplying by the appropriate sign. (Let $T_1 = T \otimes 1 \otimes \dots \otimes 1$, etc.). Write

$$D_0 = \text{Hom}_S(W, Z[T_1, T_2, \dots, T_l]).$$

Then

$$H^*(D_0) \cong H^*(S, Z[T_1, \dots, T_l])$$

and

$$H^{**}(D_0) \cong H^{**}(S, Z[[T_1, T_2, \dots, T_l]]).$$

(Here we use the fact that W may be chosen so that each W_p is finitely generated.)

Now suppose $\alpha \in H_e^{**}(C) = \prod_{n \geq 0} H^{2n}(C)$. (Generally we shall write such elements as infinite series $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n + \dots$ with $\alpha_n \in H^{2n}(C)$.) Let $a = a_0 + a_1 + \dots + a_n + \dots$ be a cocycle representing α . Then we may define a map

$$F : Z[T] \rightarrow C$$

of complexes such that $F(T^{2n}) = a_n$, $F(T^{2n+1}) = 0$, and

$$F^{**}(1/(1 - T^2)) = \alpha.$$

Moreover, if a' also represents α and F' is defined for a' , then F and F' are homotopic. Let

(2) Following [3], $\text{Hom}(X, Y) = \sum_{p,q} \text{Hom}(X_p, Y^q)$ with the obvious differentiations.

$$Y = \prod_{i=1}^l (1/(1 - T_i^2)).$$

Y is an element of $Z[[T_1, \dots, T_l]]^S$ which is naturally isomorphic to

$$H^0(S, Z[[T_1, \dots, T_l]]).$$

Define $1_S \int \alpha$ in $H_e^{**}(\text{Hom}_S(W, C^l))$ by the formula

$$1_S \int \alpha = \text{Hom}_S(1, F^l)^{**}(Y).$$

In view of the above remarks, this object is well defined, it does not depend on a , F , or even W .

We now summarize certain properties of the operation of forming $1 \int \alpha$ which will be useful below.

1. Let $S' < S$ and suppose $G: C \rightarrow C'$ is a map of complexes. Then there is an induced map

$$G^{**}: H^{**}(\text{Hom}_S(W, C^l)) \rightarrow H^{**}(\text{Hom}_{S'}(W, C'^l)).$$

(In fact, we could replace W by any S' -projective resolution of Z in the second term.) Then

$$G^{**}(1_S \int \alpha) = 1_{S'} \int G^{**}(\alpha).$$

2. Let $S_1 < \mathcal{S}_{l_1}$, $S_2 < \mathcal{S}_{l_2}$ and let W_1 and W_2 be corresponding projective resolutions of Z . As in [4, §5, p. 231], $W_1 \otimes W_2^{l_2} = W$ is an $S_1 \int S_2 = S$ -projective resolution of Z . In the obvious way define the homomorphism (of functors) ⁽³⁾.

$$\mathcal{A}: \text{Hom}_{S_1}(W_1, \text{Hom}_{S_2}(W_2, C^{l_2})^{l_1}) \rightarrow \text{Hom}_{S_1 \int S_2}(W, C^{l_1 l_2}).$$

(Here $S_1 \int S_2$ is embedded in $\mathcal{S}_{l_1 l_2}$ as in §1.) Now for $\alpha \in H^{**}(C)$, $1_{S_1} \int (1_{S_2} \int \alpha)$ makes sense as an element of the homology of the left-hand side and

$$\mathcal{A}^{**}(1_{S_1} \int (1_{S_2} \int \alpha)) = 1_{S_1 \int S_2} \int \alpha.$$

3. Let S_1 and S_2 be as above and let $l = l_1 + l_2$. Then $S_1 \times S_2$ may be imbedded in \mathcal{S}_l by letting S_1 act on the first l_1 symbols and S_2 on the remaining l_2 symbols. Also $W_1 \otimes W_2$ is an $S_1 \times S_2$ -projective resolution of Z . Again consider the obvious homomorphism⁽³⁾

$$\mathcal{B}: \text{Hom}_{S_1}(W_1, C^{l_1}) \otimes \text{Hom}_{S_2}(W_2, C^{l_2}) \rightarrow \text{Hom}_{S_1 \times S_2}(W_1 \otimes W_2, C^l).$$

Now the *external product* $1_{S_1} \int \alpha \times 1_{S_2} \int \alpha$ is an element of the homology of the left-hand side and

$$\mathcal{B}^{**}(1_{S_1} \int \alpha \times 1_{S_2} \int \alpha) = 1_{S_1 \times S_2} \int \alpha.$$

⁽³⁾ Here since C is a *complex* we must include a sign change in the definitions of \mathcal{A} and \mathcal{B} .

4. Let $S < \mathcal{S}_l$, let W be an S -projective resolution of Z and $\Delta: W \rightarrow W \otimes W$ a diagonal map. Let C_1 and C_2 be cochain complexes. Then Δ (and permuting the factors of the tensor product) induces a homomorphism

$$\text{Hom}_{S \times S}(W \otimes W, C_1^l \otimes C_2^l) \rightarrow \text{Hom}_S(W, (C_1 \otimes C_2)^l).$$

(Here $S \times S$ acts on $C_1^l \otimes C_2^l$ "componentwise" and S acts on $(C_1 \otimes C_2)^l$ as usual.) Combining this homomorphism with one defined as in (3) we get a homomorphism

$$\mathcal{P}: \text{Hom}_S(W, C_1^l) \otimes \text{Hom}_S(W, C_2^l) \rightarrow \text{Hom}_S(W, (C_1 \otimes C_2)^l).$$

Let $\alpha_i \in H_e^{**}(C_i)$ ($i = 1, 2$). Then $1_S \int \alpha_1 \times 1_S \int \alpha_2$ is in the homology of the left-hand side, $\alpha_1 \times \alpha_2 \in H_e^{**}(C_1 \otimes C_2)$, and

$$\mathcal{P}^{**}(1_S \int \alpha_1 \times 1_S \int \alpha_2) = 1_S \int \alpha_1 \times \alpha_2.$$

If in addition $C_1 = C_2 = C$ and C is a differential graded ring, then using \mathcal{P} , we can make $\text{Hom}_S(W, C^l)$ such a ring and

$$(1_S \int \alpha_1)(1_S \int \alpha_2) = 1_S \int \alpha_1 \alpha_2.$$

5. To verify the above statements or generally to study $1_S \int \alpha$ it is convenient to reduce things to the cochain level. Let a represent $\alpha \in H_e^{**}(C)$ as above. Let $\varepsilon: W \rightarrow Z$ be the augmentation for the resolution W . Define the cocycle $\varepsilon \int a \in \text{Hom}_S(W, C^l)$ by

$$(\varepsilon \int a)(w) = \varepsilon(w)(a \otimes a \otimes \cdots \otimes a).$$

Then $\varepsilon \int a$ represents $1_S \int \alpha$.

5. **The norm map.** Let H be a group and suppose $S < \mathcal{S}_l$, etc., as above. Take $C = \text{Hom}_H(X, A)$ where X is an H -projective resolution of Z and A is an H -module. Then $H^{**}(C) = H^{**}(H, A)$ and we wish to extend the definition of §4 so as to define an element $1_S \int \alpha$ (using the *same notation*) in $H^{**}(S \int H, A^l)$. As in §4, we have the homomorphism

$$\mathcal{A}_1: \text{Hom}_S(W, \text{Hom}_H(X, A)^l) \rightarrow \text{Hom}_{S \int H}(W \otimes X^l, A^l).$$

Define $1_S \int \alpha$ (for $\alpha \in H_e^{**}(H, A)$) in the homology of the right-hand side as $\mathcal{A}_1^{**}(1_S \int \alpha)$. Since \mathcal{A}_1 is a homomorphism even of functors the definition is well made. The facts outlined in §4 carry over as follows:

1. If $\phi: H' \rightarrow H$ is a homomorphism of groups, $f: A \rightarrow A'$ a consistent homomorphism of H -modules we have ($i: S' < S$)

$$(i \int \phi)^{**}(1_{S'} \int \alpha) = 1_{S'} \int \phi^{**}(\alpha).$$

2. We have an isomorphism of groups

$$\mathcal{A}_2 : (S_1 \int S_2) \int H \xrightarrow{\cong} S_1 \int (S_2 \int H)$$

and

$$\mathcal{A}_2^{**}(1_{S_1} \int (1_{S_2} \int \alpha)) = 1_{S_1} \int S_2 \int \alpha.$$

3. We have an isomorphism

$$\mathcal{B}_2 : S_1 \times S_2 \int H \xrightarrow{\cong} S_1 \int H \times S_2 \int H$$

and

$$\mathcal{B}_2^{**}(1_{S_1} \int \alpha \times 1_{S_2} \int \alpha) = 1_{S_1 \times S_2} \int \alpha.$$

4. Let H_1 and H_2 be groups and A_1, A_2 corresponding modules over H_1, H_2 . The diagonal monomorphism $S \rightarrow S \times S$ induces a monomorphism

$$\mathcal{P}_2 : S \int H_1 \times H_2 \rightarrow S \int H_1 \times S \int H_2.$$

\mathcal{P}_2 induces

$$\mathcal{P}_2^{**} : H^{**}(S \int H_1 \times S \int H_2, A_1^l \otimes A_2^l) \rightarrow H^{**}(S \int H_1 \times H_2, (A_1 \otimes A_2)^l)$$

and

$$\mathcal{P}_2^{**}(1_S \int \alpha_1 \times 1_S \int \alpha_2) = 1_S \int \alpha_1 \times \alpha_2.$$

If $H_1 = H_2 = H$ and $A_1 \otimes A_2 \rightarrow A_3$ is an H pairing, then there is a corresponding $S \int H$ pairing $A_1^l \otimes A_2^l \rightarrow A_3^l$ and using the cup product with respect to these pairings we have

$$(2) \quad 1_S \int \alpha_1 \cup 1_S \int \alpha_2 = 1_S \int \alpha_1 \cup \alpha_2.$$

In case $A_1 = A_2 = A_3 = k$ is an H -ring (with the product pairing), formula (2) applies. In addition if k is commutative and H acts trivially the product mapping $k^l \rightarrow k$ is a homomorphism of $S \int H$ rings (where $S \int H$ acts trivially on k) and we may use this map to carry $1_S \int \alpha$ to an element $H^{**}(S \int H, k)$. Using the same notation for this new element, we see that formula (2) still applies (and in fact all the above formulas).

5. If $a \in \text{Hom}_H(X, A)$ represents α , then $\varepsilon \otimes a^l \in \text{Hom}_{S \int H}(W \otimes X^l, A^l)$ represents $1_S \int \alpha$.

Let G be a group, H a subgroup of finite index l . Let $\Phi : G \rightarrow \mathcal{S}_l \int H$ be defined with respect to some transversal. For $\alpha \in H_e^{**}(H, A)$ we define $\mathcal{N}_{H \rightarrow G}(\alpha) \in H_e^{**}(G, \mathcal{M}_{H \rightarrow G}(A))$ by

$$\mathcal{N}_{H \rightarrow G}(\alpha) = \Phi^{**}(1_{\mathcal{S}_l} \int \alpha).$$

Notice that \mathcal{S}_l may be replaced by the transitive subgroup S which is the image of the composite homomorphism

$$G \rightarrow \mathcal{S}_l \int H \xrightarrow{\text{surjection}} \mathcal{S}_l.$$

(This is the subgroup of permutations of the coset space G/H produced by the action of G .)

If $\Phi' : G \rightarrow \mathcal{S}_l \wr H$ is defined with respect to another transversal and $\mathcal{N}'_{H \rightarrow G}$ is defined by Φ' , then

$$\mathcal{N}_{H \rightarrow G}(\alpha) = \mu^{**} \mathcal{N}'_{H \rightarrow G}(\alpha).$$

Thus $\mathcal{N}_{H \rightarrow G}(\alpha)$ is essentially well defined.

If k is a commutative ring on which G acts, then $\mathcal{M}_{H \rightarrow G}(k)$ becomes a G -ring (treating k as an H -ring), and $\psi : \mathcal{M}_{H \rightarrow G}(k) \rightarrow k$ defined by

$$\psi(a_1 \otimes a_2 \otimes \cdots \otimes a_l) = \tau_1 a_1 \cdot \tau_2 a_2 \cdot \cdots \cdot \tau_l a_l$$

(where τ_1, \dots, τ_l is the same transversal as that in the definition of $\mathcal{M}_{H \rightarrow G}(k)$) is a homomorphism of G -rings.

Given $\alpha \in H_e^{**}(H, k)$, define $\mathcal{N}_{H \rightarrow G}(\alpha) \in H_e^{**}(G, k)$ as $\psi^{**} \mathcal{N}_{H \rightarrow G}(\alpha)$ (again using the same notation). With coefficients in k there is no ambiguity in the module and no ambiguity in the definition of $\mathcal{N}_{H \rightarrow G}$.

REMARK. Notice the fact that since the group G is smaller than $\mathcal{S}_l \wr H$, it is possible to drop the restriction of trivial action required in the more general situation.

6. Formal properties. In what follows it will be possible to interpret the formal rules in different contexts depending on whether the module is general or a commutative ring on which the group G acts. *The contention always will be that the formula in question is valid in all such contexts.*

Since the module $\mathcal{M}_{H \rightarrow G}(A)$ is not well defined except up to isomorphism there will be some ambiguity for the general situation. We shall ignore this ambiguity and write “=” where there is only a canonical isomorphism derived from an element μ as in §§2 and 3. We can always insure true equality by choosing transversals consistently.

Finally we assume always that $\alpha \in H_e^{**}(H, A)$, etc.

1. *Transitivity.* Let $G > K > H$ where H is of finite index l . Let $l_1 = (G : K)$, $l_2 = (K : H)$ so that $l = l_1 l_2$. Let $\tau'_1, \tau'_2, \dots, \tau'_{l_1}$ be a transversal of K in G , $\tau''_1, \tau''_2, \dots, \tau''_{l_2}$ a transversal of H in K . Then the elements $\tau'_i \tau''_j$ (ordered lexicographically) give a transversal of H in G . Let A be a H -module. Then

$$\mathcal{M}_{H \rightarrow G}(A) = \mathcal{M}_{K \rightarrow G}(\mathcal{M}_{H \rightarrow K}(A)).$$

PROPOSITION 1. *Let $G > K > H$ be as above.*

$$\mathcal{N}_{H \rightarrow G}(\alpha) = \mathcal{N}_{K \rightarrow G}(\mathcal{N}_{H \rightarrow K}(\alpha)).$$

Proof. Let $\Phi_1 : G \rightarrow \mathcal{S}_{l_1} \wr K$, $\Phi_2 : K \rightarrow \mathcal{S}_{l_2} \wr H$, $\Phi : G \rightarrow \mathcal{S}_l \wr H$ be defined with respect to the above transversals. We have a commutative diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\Phi} & \mathcal{S}_i \int H \\
 \downarrow \Phi_1 & & \uparrow \text{inclusion} \\
 & & (\mathcal{S}_{l_1} \int \mathcal{S}_{l_2}) \int H \\
 & & \uparrow \mathcal{A}_2^{-1} \\
 \int \mathcal{S}_{l_1} K & \xrightarrow{1 \int \Phi_2} & \mathcal{S}_{l_1} \int (\mathcal{S}_{l_2} \int H).
 \end{array}$$

The proposition now follows by applying rules 5-1 and 5-2.

2. *Double coset rule.* Let G be a group and H and K subgroups of finite index. Let $G = \bigcup_{i=1}^d K\tau_i H$ be a decomposition of G into disjoint double cosets. Let $K\tau_i H = \bigcup_{j=1}^{l_i} \rho_{ij}\tau_i H$ be a decomposition into disjoint left cosets where $\rho_{i1}, \rho_{i2}, \dots, \rho_{il_i}$ is a left transversal of $\tau_i H\tau_i^{-1} \cap K$ in K . (For convenience write $\tau_i H\tau_i^{-1} = H_i$.) Order the elements $\rho_{ij}\tau_i$ lexicographically and define $\Phi : G \rightarrow \mathcal{S}_i \int H$ with respect to this transversal of H in G . Also define $\Phi_i : K \rightarrow \mathcal{S}_{l_i} \int H_i \cap K$ with respect to the appropriate transversal. Let $c_i : \tau_i H\tau_i^{-1} \rightarrow H$ be the map $\tau\sigma\tau^{-1} \rightarrow \sigma$.

If A is an H -module, then it may be viewed as an H_i -module A_i through c_i and as an $H_i \cap K$ -module by restriction. Then as K -modules

$$\mathcal{M}_{H \rightarrow G}(A) = \mathcal{M}_{H_1 \cap K \rightarrow K}(A_1) \otimes \mathcal{M}_{H_2 \cap K \rightarrow K}(A_2) \otimes \cdots \otimes \mathcal{M}_{H_d \cap K \rightarrow K}(A_d)$$

where K acts diagonally on the right.

PROPOSITION 2. *With notation as above,*

$$\text{res}_{G \rightarrow K} \mathcal{N}_{H \rightarrow G}(\alpha) = \prod_{i=1}^d \mathcal{N}_{H_i \cap K \rightarrow K}(\text{res}_{H_i \rightarrow H_i \cap K} c_{\tau_i}^{**}(\alpha)).$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
 \mathcal{S}_i \int H & \xleftarrow{\text{inclusion}} & (\mathcal{S}_{l_1} \times \mathcal{S}_{l_2} \times \cdots \times \mathcal{S}_{l_d}) \int H \\
 \uparrow \Phi & & \uparrow \mathcal{B}_2^{-1} \\
 G & & \mathcal{S}_{l_1} \int H \times \mathcal{S}_{l_2} \int H \times \cdots \times \mathcal{S}_{l_d} \int H \\
 \uparrow \text{inclusion} & & \uparrow 1 \int c_{\tau_1} \times \cdots \times 1 \int c_{\tau_d} \\
 K & & \mathcal{S}_{l_1} \int H_1 \times \mathcal{S}_{l_2} \int H_2 \times \cdots \times \mathcal{S}_{l_d} \int H_d \\
 \searrow \text{diagonal} & & \nearrow 1 \int \text{incl.} \times \cdots \times 1 \int \text{incl.} \\
 K \times K \times \cdots \times K & & \\
 \Phi_1 \times \cdots \times \Phi_d & \searrow & \mathcal{S}_{l_1} \int H_1 \cap K \times \cdots \times \mathcal{S}_{l_d} \int H_d \cap K.
 \end{array}$$

The result follows by applying rules 5-1 and 5-3 and the definition of cup product in the cohomology of K .

3. *Automorphism property.* Let $\phi: G' \rightarrow G$ be a homomorphism of groups; let H be a subgroup of finite index l in G and let $H' = \phi^{-1}(H)$. Suppose H' is also of index l in G' . If A, A' are H, H' modules respectively and $f: A \rightarrow A'$ is a homomorphism consistent with $\phi_0 = \phi|_H$, then f induces a homomorphism

$$\mathcal{M}(f): \mathcal{M}_{H \rightarrow G}(A) \rightarrow \mathcal{M}_{H' \rightarrow G'}(A')$$

consistent with ϕ .

PROPOSITION 3. Let $\phi: G' \rightarrow G$ be as above with $(G:H) = (G':H') = l$ (where $H' = \phi^{-1}(H)$). Then

$$\phi^{**}(\mathcal{N}_{H \rightarrow G}(\alpha)) = \mathcal{N}_{H' \rightarrow G'}(\phi_0^{**}(\alpha)).$$

Proof. Let $\tau'_1, \tau'_2, \dots, \tau'_l$ be a transversal of H' in G' ; then $\tau_1 = \phi(\tau'_1), \dots, \tau_l = \phi(\tau'_l)$ is a transversal of H in G . Define Φ' and Φ with respect to these transversals. Consider the commutative diagram

$$\begin{array}{ccc} G' & \xrightarrow{\Phi'} & \mathcal{S}_l \int H' \\ \downarrow \phi & & \downarrow 1 \int \phi_0 \\ G & \xrightarrow{\Phi} & \mathcal{S}_l \int H. \end{array}$$

The result follows from 5-1.

4. *Multiplicative property.* Let H be a subgroup of G of finite index. If $A_1 \otimes A_2 \rightarrow A_3$ is a pairing of H -modules it induces a pairing of G -modules

$$\mathcal{M}_{H \rightarrow G}(A_1) \otimes \mathcal{M}_{H \rightarrow G}(A_2) \rightarrow \mathcal{M}_{H \rightarrow G}(A_3).$$

PROPOSITION 4. With $G > H$ as above,

$$\mathcal{N}_{H \rightarrow G}(\alpha \cup \beta) = \mathcal{N}_{H \rightarrow G}(\alpha) \cup \mathcal{N}_{H \rightarrow G}(\beta).$$

Proof. Let $\Phi: G \rightarrow \mathcal{S}_l \int H$ be defined with respect to a transversal. Since any homomorphism of groups induces a multiplicative homomorphism of cohomology, the result follows by applying Φ^{**} to the formula (2) of 5-4.

REMARK 1. Since $\mathcal{N}_{H \rightarrow G}$ is multiplicative it defines a homomorphism $\mathcal{N}: U_e(H) \rightarrow U_e(G)$ from the group of units $U_e(H)$ of the ring $H_e^{**}(H, Z)$ to the corresponding group for G .

REMARK 2(⁴). It is natural to investigate the quantity $\mathcal{N}_{H \rightarrow G}(\text{res}_{G \rightarrow H} \alpha)$ where for example $\alpha \in H^{2n}(G, Z)$. If the analogy with transfer held true then this quantity would equal α^l . In fact except in special cases this equality is false.

(⁴) I should like to thank R. G. Swan for pointing out the substance of this remark to me and also for much valuable advice concerning the contents of this paper.

For example let $G = P \times H$ where P is cyclic of prime order. Let $\delta: H \rightarrow H^P$ be the diagonal homomorphism; then the map $\Phi: G \rightarrow P \wr H$ may be chosen to be the obvious induced homomorphism $P \times H \rightarrow P \cdot H^P$. Let X be an H -projective resolution of Z and $\Delta: X \rightarrow X^P$ a diagonal map over δ . If it were possible to choose Δ to be a P -homomorphism the questioned equality would hold. Unfortunately we know that this is generally not possible, and this fact is responsible for the existence of the *Steenrod operations*. In fact one can convince oneself that all the Steenrod reduced powers except $u \rightarrow u^P$ would be trivial were the equality in question to hold. (Using the properties of the Steenrod operations, we can show that for $\lambda \in H^2(H, Z_p)$, $\mathcal{N}(\lambda) \in H^{2P}(P \times H, Z_p)$ is given by

$$1 \times \lambda^P - \mu^{P-1} \times \lambda$$

where μ generates $H^2(P, Z_p)$. It is possible also to derive this formula by means of Chern classes.)

7. Applications. Let G be a group and H a subgroup of finite index. Restrict attention to commutative G -rings k . Notice that this is no essential restriction since if A is a G -module, $k = Z \oplus A$ can be made a commutative G -ring in which $A^2 = 0$ in the usual way.

THEOREM 1. Let G be a group, H a subgroup of finite index l and k a commutative G -ring. For $\chi \in H^2(H, k)$ we have

$$\mathcal{N}_{H \rightarrow G}(1 + \chi) = \mathcal{N}_0(\chi) + \mathcal{N}_1(\chi) + \cdots + \mathcal{N}_l(\chi)$$

($\mathcal{N}_i(\chi) \in H^{2i}(G, k)$) where $\mathcal{N}_0(\chi) = 1$, $\mathcal{N}_1(\chi) = \text{tr}_{H \rightarrow G}(\chi)$, and $\mathcal{N}_l(\chi) = \mathcal{N}_{H \rightarrow G}(\chi)$.

Proof. Let $\Phi: G \rightarrow \mathcal{S}_l \wr H$ be defined with respect to some transversal. If W is an \mathcal{S}_l -resolution and X an H -resolution of Z , then $W \otimes X^l$ is an $\mathcal{S}_l \wr H$ resolution and through Φ a G -projective resolution of Z . Let $\mu: X_0 \rightarrow Z$, $\varepsilon: W_0 \rightarrow Z$ be the augmentations for these resolutions. Suppose the transversal defining Φ has been chosen so that $\tau_1 = 1$. Then

$$F = \varepsilon \otimes 1_X \otimes \mu^{l-1} : W \otimes X \otimes X^{l-1} \rightarrow X$$

is a homomorphism of H -projective resolutions of Z and induces the identity in cohomology for H .

Let $\eta: Z \rightarrow k$ be defined by $\eta(1) = 1_k$. Then $\mu': X_0 \rightarrow k$ defined by $\mu' = \eta \circ \mu$ represents $1 \in H^0(H, k)$. Let $f: X_{2r} \rightarrow k$ represent $\chi \in H^{2r}(H, k)$. Then in view of 5-5 (and the definition of the map $\psi: k \xrightarrow{l} k$) $\mathcal{N}(1 + \chi)$ is represented by $\mathcal{N}(\mu' + f)$ where

$$\mathcal{N}(g)(w_0 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_l) = \varepsilon'(w_0)(\tau_1 g(x_1))(\tau_2 g(x_2)) \cdots (\tau_l g(x_l)).$$

The only statement in Theorem 1 which needs more discussion is that concerning transfer. $\mathcal{N}_1(\chi)$ is represented by $\mathcal{N}_1(f)$ where

$$\begin{aligned}
\mathcal{N}_1(f)(w_0 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_l) \\
&= \sum_{i=1}^l \varepsilon'(w_0) \mu'(x_1) \mu'(x_2) \cdots (\tau_i f(x_i)) \cdots \mu'(x_l) \\
&= \sum_{i=1}^l \tau_i \tilde{f}(\tau_i^{-1}(w_0 \otimes x_1 \otimes \cdots \otimes x_l)),
\end{aligned}$$

or $\mathcal{N}_1(f) = \text{tr}_{H \rightarrow G} \tilde{f}$ where

$$\tilde{f}(w_0 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_l) = \varepsilon'(w_0) f(x_1) \mu'(x_2) \cdots \mu'(x_l).$$

(Remember $\tau_1 = 1!$) However $F^*(f) = \tilde{f}$ so that \tilde{f} represents χ as required.

THEOREM 2. *Let G be a group, H a normal subgroup of finite index; let k be a commutative G -ring. Then for $\alpha \in H_e^{**}(H, k)$,*

$$c_{\tau_1}(\alpha) \cdot c_{\tau_2}(\alpha) \cdots c_{\tau_l}(\alpha) \in \text{Im } \text{res}_{G \rightarrow H}$$

(where τ_1, \dots, τ_l is a transversal of H in G).

Proof. Clear from the double coset rule.

THEOREM 3. *Let G be a finite group and H a subgroup of G . There is an integer $m \mid (G:1)$ such that*

$$\text{res} : H^{2q}(G, \mathbb{Z}) \rightarrow H^{2q}(H, \mathbb{Z})$$

is nontrivial for $q \equiv 0 \pmod{m}$.

Proof. It is enough to prove this for H cyclic of prime order. Let χ generate $H^2(H, \mathbb{Z})$. By the double coset rule

$$\text{res}_{G \rightarrow H} \mathcal{N}_{H \rightarrow G}(1 + \chi) = \prod_{i=1}^d \mathcal{N}_{H_i \cap H \rightarrow H} \text{res}_{H_i \rightarrow H_i \cap H}(1 + c_{\tau_i} \chi)$$

where $G = \bigcup_{i=1}^d H \tau_i H$ and $H_i = \tau_i H \tau_i^{-1}$.

However,

$$H_i \cap H = \begin{cases} H & \tau_i \in \text{Normalizer of } H = N, \\ \{1\} & \text{otherwise.} \end{cases}$$

Since $\mathcal{N}_{\{1\} \rightarrow H}(1 + 0) = 1$, we can ignore all terms in the product where $\tau_i \notin N$. Moreover, in N , double cosets are all single cosets. Hence

$$\text{res}_{G \rightarrow H} \mathcal{N}_{H \rightarrow G}(1 + \chi) = (1 + c_{\tau_1} \chi)(1 + c_{\tau_2} \chi) \cdots (1 + c_{\tau_m} \chi)$$

where $m = (N:H) \mid (G:1)$. Since the term $c_{\tau_1} \chi \cdots c_{\tau_m} \chi \neq 0$, Theorem 3 follows.

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